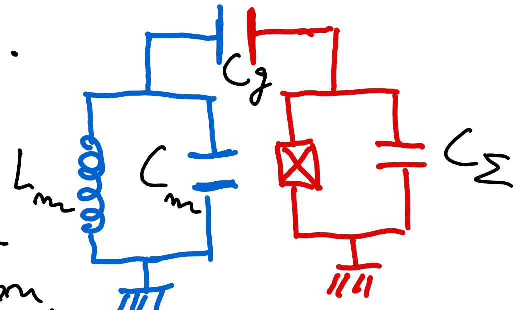
Ligth-Matter Interaction in Circuit QED

Having introduced the quantum harmonic oscillator and the transmon artificial atom, we can now consider their interaction. Because of their large size coming from the requirement of having a low charging energy, transmon qubits can very naturally be capacitively coupled to microwave resonators.



With the resonator taking place of the classical voltage source V_g, capacitive coupling to a resonator can be introduced in the transmon Hamiltonian with a quantized voltage $n_g \to -\hat{n}_r$, representing the charge bias of the transmon due to the resonator (the choice of sign is simply a common convention).

The Hamiltonian of the combined systems is:

$$\hat{H} = 4E_c(\hat{n}+\hat{n}_r)^2 - E_J\cos\hat{\varphi} - \sum_m \hbar\omega_m \hat{a}_m^\dagger \hat{a}_m ,$$

where $\hat{n}_r = \sum_m \hat{n}_m$ with $\hat{n}_m = (C_g/C_m)\hat{Q}_m/2e$ the contribution to the charge bias due to the mth resonator mode. 2

Here C_g is the coupling capacitance and C_m the associated resonator mode capacitance. Here we assumed $C_g \ll C_\Sigma, C_m$.

Assuming that the transmon frequency is much closer to one of the resonator modes than all the other modes, $|\omega_0 - \omega_q| \ll |\omega_m - \omega_q|$ for $m \geq 1$, we truncate the sum over m to a single term.

In this single mode approximation, the Hamiltonian reduces to a single oscillator of frequency ω_r coupled to a transmon.

Using the creation and annihilation operators, we have

$$\hat{H} \simeq \hbar\omega_r \hat{a}^\dagger \hat{a} + \hbar\omega_q \hat{b}^\dagger \hat{b} - \frac{E_c}{2} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} - \hbar g (\hat{b}^\dagger - \hat{b})(\hat{a}^\dagger - \hat{a}),$$

where $g = \omega_r \dfrac{C_g}{C_\Sigma} \left(\dfrac{E_J}{2E_c}\right)^{1/4} \sqrt{\dfrac{\pi Z_r}{R_K}}$ the oscillator-transmon coupling constant.

Here, $Z_r = \sqrt{\frac{\mathcal{L}_r}{\mathcal{C}_r}}$ is the characteristic impedance of the resonator having an inductance per unit length $\mathcal{L}_r$ and a capacitance per unit length $\mathcal{C}_r$. $R_K = h/e^2 \sim 25.8\,k\Omega$ is the resistance quantum. 3

The above Hamiltonian can be further simplified in the experimentally valid situation where the coupling is much smaller than the system frequencies $|g| \ll \omega_r, \omega_q$. Invoking the rotating-wave-approximation

$$\hat{H} \approx \hbar\omega_r \hat{a}^\dagger \hat{a} + \hbar\omega_q \hat{b}^\dagger \hat{b} - \frac{E_C}{2} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hbar g \left(\hat{b}^\dagger \hat{a} + \hat{b} \hat{a}^\dagger \right).$$

By introducing a length scale "l" corresponding to the distance a Cooper pair travels when tunneling across the transmon's junction, we can interpret $\hbar g = d_0 \mathcal{E}_0$ with $d_0 = 2el(E_J/32E_C)^{1/4}$ the dipole moment of the transmon and $\mathcal{E}_0 = (\omega_r/l)(C_g/C_\Sigma)\sqrt{\hbar Z_r/2}$ the resonator's zero-point electric field seen by the transmon.

Since these 2 factors can be made very large ($d_e \gg 2el$) the electric-dipole interaction strength g can be made very large, much more than with natural atoms in cavities.

We can also express $g = \omega_r \frac{C_g}{C_\Sigma} \left(\frac{E_J}{2E_c}\right)^{1/4} \sqrt{\frac{Z_r}{Z_{vac}}} \sqrt{2\pi\alpha}$

where $\alpha = Z_{vac}/2R_K$ is the fine structure constant and $Z_{vac} = \sqrt{\mu_0/\varepsilon_0} \sim 377\,\Omega$ the impedance of the vacuum with ε_0 (μ_0) the vacuum permittivity (permeability).

Very large couplings can be achieved by working with large values of $E_J/E_c \Rightarrow$ transmon regime. Large g is therefore obtained at the expenses of reducing the transmon's relative anharmonicity $-E_c/\hbar\omega_q \simeq -\sqrt{E_c/8E_J}$.

The coupling can be also boosted by increasing the resonator impedance $g \propto \sqrt{Z_r}$

Jaynes-Cummings model:

The qubit Hamiltonian $\hat{H}_q = -\frac{\omega_q}{2}\sigma_z$ has two eigenstates $\{|g\rangle, |e\rangle\}$ corresponding to two eigenvalues $\{\mp \omega_q/2\}$. Similarly, a single cavity mode Hamiltonian presents an infinite number of eigenstates $\{|n\rangle\}$ with eigenvalues $\{\omega_c(n+1/2)\}$ corresponding to n photons in that mode.

Here we are interested to know what are the eigenstates and eigenvalues of the hybrid system of the cavity and the qubit combined via an interaction Hamiltonian

$$H_{Rabi} = \omega_c\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) - \frac{1}{2}\omega_q\sigma_z - g\left(\hat{a}+\hat{a}^+\right)\left(\sigma_- + \sigma_+\right)$$

In the case there is not interaction between qubit and cavity ($g=0$) the eigenstates of the qubit-cavity system are simply the tensor product of the cavity and the qubit eigenstates $\{|g\rangle|n\rangle, |e\rangle|n\rangle\}$.

Those are called the "bare" states and the correspon=ding eigenvalues are simply the sum of the eigenvalues for each qubit and cavity eigenstates $\left\{\pm\omega_q/2 + \omega_c\left(n+\frac{1}{2}\right)\right\}$.

$|g\rangle|0\rangle \rightarrow$ qubit in the ground state, no photon in the cavity

$|g\rangle|n+1\rangle \longrightarrow$ qubit in the ground state, n+1 photons in the cavity

$|e\rangle|n\rangle \rightarrow$ qubit in the excited state, n photons in the cavity

When the qubit and the cavity interact ($g \neq 0$), bare states no longer are the energy eigenstates for the system. Yet, we can represent the total Hamiltonian in the bare basis and attempt to diagonalize it to find its eigenstates and eigenvalues. Before we do this, we simplify the interac=tion Hamiltonian by the rotating wave approximation. This approximation is valid in most practical situations where the coupling strength is much less than both, the qubit, and cavity frequencies, $g \ll \omega_q, \omega_c$, and also

$|\omega_c - \omega_q| << |\omega_c + \omega_q|$.

Having this situation in mind, let's revisit the interaction Hamiltonian where we have four terms

$$H_{int} \Rightarrow \underline{\hat{a}^{+}\sigma_- + \hat{a}\,\sigma_+} + \hat{a}^{+}\sigma_+ + \hat{a}\,\sigma_-$$

The first term describes the decay of the qubit and creation of a photon for the cavity and second term accounts for an excitation of the qubit and annihilation of a photon in the cavity. These processes somehow "conserve" the total energy in the system since the energy change would be $\pm(\omega_c - \omega_q)$, which is much less than the total energy in the system even in the few photon regime where $E_{tot} \sim \omega_c + \omega_q$. However, the last two terms correspond to the excitation (decay) of the qubit and creation (annihilation) of a photon for cavity which requires a relative substantial energy change $\pm(\omega_c + \omega_q)$ in the system, especially when we have only a few photons in the system. So we can simply ignore those terms.

With this rotating wave approximation, we obtain

$$H_{JC} = \omega_c\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) - \frac{1}{2}\omega_q\sigma_z - g\left(\hat{a}^{\dagger}\sigma_- + \hat{a}\,\sigma_+\right)$$

called the Jaynes-Cummings Hamiltonian.

Although the RWA simplifies the Hamiltonian, still we have to deal with an infinite dimentional Hilbert space (since the number of photons n ranges from $0 \to \infty$), which means the Hamiltonian is a semi-infinite matrix. If we use the bare basis to represent the H_{JC} in the form of matrix we find,

$$H_{JC} = \begin{pmatrix}
\frac{1}{2}\omega_c - \frac{\omega_q}{2} & 0 & 0 & 0 & & \\
0 & \frac{3}{2}\omega_c - \frac{\omega_q}{2} & g & 0 & & 0 \\
 & g & \frac{3}{2}\omega_c + \frac{\omega_q}{2} & 0 & & \\
 & & & \cdots & & \\
 & 0 & & 0 & \left(n+\frac{1}{2}\right)\omega_c - \frac{\omega_q}{2} & \sqrt{n+1}\,g \\
 & & & 0 & \sqrt{n+1}\,g & \left(n+\frac{1}{2}\right)\omega_c + \frac{\omega_q}{2}
\end{pmatrix}$$

This Hamiltonian is block-diagonal and all blocks follow a general form. Having a block diagonal Hamiltonian makes it easy to find its eigenstates. We only need to diagonalize individual blocks and the resulting eigenvalues of each block are indeed the eigenvalues of the entire Hamiltonian.

For each block we have,

$$M_n = \begin{bmatrix} (n+\frac{1}{2})\omega_c - \frac{\omega_q}{2} & \sqrt{n+1}\, g \\ \sqrt{n+1}\, g & (n+\frac{1}{2})\omega_c + \frac{\omega_q}{2} \end{bmatrix}$$

$|g\rangle|0\rangle$ corresponding to $n=0$, and the eigenstates of M_n form a complete set of eigenstates for the entire qubit-cavity system. For the eigenvalues we have:

$$E_g = -\Delta/2$$

$$E_{\mp} = (n+1)\omega_c \mp \frac{1}{2}\sqrt{4g^2(n+1)+\Delta^2} \quad , \Delta = \omega_q - \omega_c .$$

The eigen-states associated with each of these eigen-values are called the "dressed states" of the qubit and the cavity

$$|0,-\rangle = |g\rangle|0\rangle$$

$$|n,-\rangle = \cos(\theta_n)|g\rangle|n+1\rangle - \sin(\theta_n)|e\rangle|n\rangle$$

$$|n,+\rangle = \sin(\theta_n)|g\rangle|n+1\rangle + \cos(\theta_n)|e\rangle|n\rangle$$

where $\theta_n = \frac{1}{2}\tan^{-1}(2g\sqrt{n+1}/\Delta)$ which quantifies the "level of hybridization".

In the limit of $\Delta \to 0$ where the qubit and the cavity have the same energy we have $\theta_n = \pi/4$ and the dressed states are in maximum hybridization

$$|n,\mp\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle|n+1\rangle \mp |e\rangle|n\rangle\right),$$

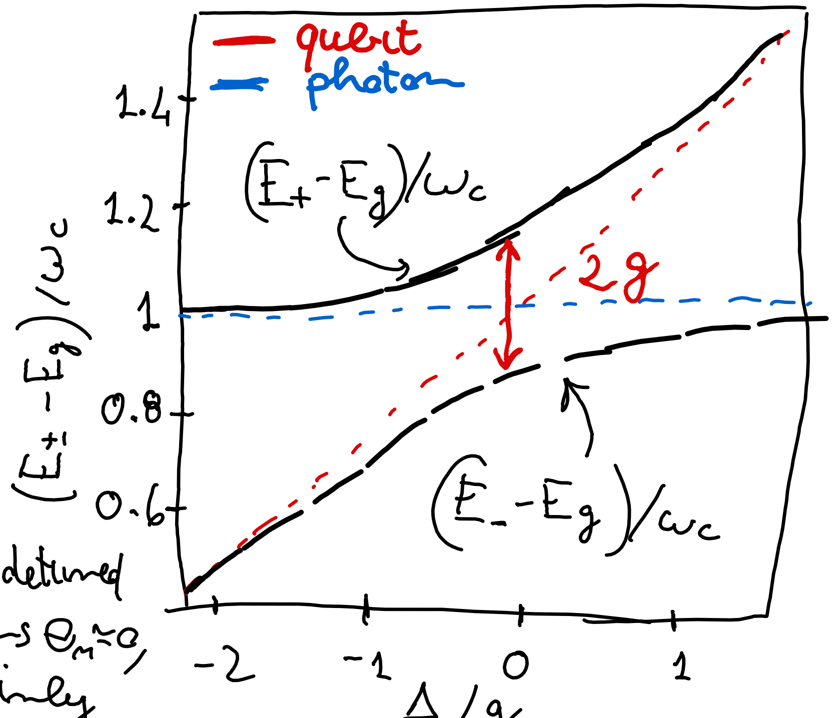
which means each of the dressed states has a 50-50% characteristic of cavity photon and qubit excitations. These states are called "polaritons". The energy difference between the first two polariton states is $2g$.

It is convenient to plot transition energy versus frequency detuning $\Delta = \omega_q - \omega_c$, since we normally characterize the system by measuring the transition frequencies by doing spectroscopy.

For example, when $n=0$ we have

$$E_\mp - E_g = \omega_c \mp \frac{1}{2}\sqrt{4g^2+\Delta^2} + \frac{\Delta}{2}$$

In Figure we plot $(E_\pm - E_g)$ versus detuning Δ, which clearly shows an avoided crossing.



If the qubit-cavity are far detuned ($\Delta \ll 0$, $\Delta \gg 0$), which means $\theta_n \approx 0$, the dressed states stay mainly either qubit or cavity-like. For $\Delta \to 0$, the dressed states start to push each other, and deviate from the corresponding bare states.

Dispersive approximation:

We can perform another approximation to the interaction Hamiltonian, valid in the regime where the cavity and qubit are far detuned $\Delta \gg g$. In such situation, the interaction is relatively weak. In this regime, the cavity and the qubit do not directly exchange energy unlike what we have in the interaction term in the JC Hamiltonian.

Let's consider the unitary transformation $\hat{T} = e^{\lambda(\sigma_- a^\dagger - a_+ a)}$ where $\lambda = g/\Delta$. If we apply this transformation to the JC Hamiltonian and keeping all terms to order λ^2

$$\hat{T} H_{JC} \hat{T}^\dagger = \omega_c\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) - \frac{1}{2}\omega_q \sigma_z - \frac{g^2}{\Delta}\hat{a}^\dagger \hat{a}\,\sigma_z + \frac{g^2}{2\Delta}\sigma_z$$

We may ignore constant terms since these do not affect dynamics, and obtain

$$\hat{H}_{disp.} \simeq \omega_c \hat{a}^\dagger \hat{a} - \frac{1}{2}\omega_q \sigma_z - \frac{g^2}{\Delta}\hat{a}^\dagger \hat{a}\,\sigma_z$$

The dispersive Hamiltonian describes the situation where the cavity and the qubit are far detuned, (13)
, the coupling is weak and dressed states are almost overlapping with the bare states.
Yet, there is a very small interaction as described by the last term in $\hat{H}_{disp.} \sim \frac{g^2}{\Delta} \hat{a}^\dagger \hat{a} \sigma_z$
In order to make a better sense of this interaction we re-arrange the terms in the Hamiltonian, as follows:

$$\hat{H}_{disp.} = (\omega_c - \chi \sigma_z) \hat{a}^\dagger \hat{a} - \frac{1}{2} \omega_q \sigma_z , \quad \chi = g^2/\Delta .$$

$\boxed{\chi \text{ is the dispersive shift}}$.

We see that the dispersive interaction is manifested as a qubit-state-dependent shift for the cavity. If the qubit is in the ground (excited) state $|g\rangle$ ($|e\rangle$), then $\langle \sigma_z \rangle = 1 \; (= -1)$, which means that the cavity frequency shifts by $+(-)\chi$.
Therefore, one can detect this frequency shift for the cavity to determine the state of the qubit.